

Decomposition of intra-regular po - Γ -semigroups into simple components

Niovi Kehayopulu

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Abstract We keep the definition of intra-regularity (left regularity) of po - Γ -semigroups introduced in [6] which is absolutely necessary for the investigation. Being able to describe the form of the elements of the principal filter by using this definition, we study the decomposition of an intra-regular po - Γ -semigroup into simple components. Then we prove that a po - Γ -semigroup M is intra-regular and the ideals of M form a chain if and only if M is a chain of simple semigroups. Moreover, a po - Γ -semigroup M is intra-regular and the ideals of M form a chain if and only if the ideals of M are prime. Finally, for an intra-regular po - Γ -semigroup M , the set $\{(x)_N \mid x \in M\}$ coincides with the set of all maximal simple subsemigroups of M . A decomposition of left regular and left duo po - Γ -semigroup into left simple components has been also given.

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1 Introduction and prerequisites

An ordered Γ -semigroup (shortly po - Γ -semigroup) is a nonempty set M with a set of binary operations Γ on M and an order relation on M such that $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for every $c \in M$ and every $\gamma \in \Gamma$. For a po - Γ -semigroup M and a subset H of M we denote by $(H]$ the subset of M defined by $(H] = \{t \in M \mid t \leq a \text{ for some } a \in H\}$. We have $M = (M]$, and for any two subsets A, B of M , we have $A \subseteq (A]$; if A is a right (or left) ideal of M , then $A = (A]$; $(A]\Gamma(B] \subseteq (A\Gamma B]$; $A \subseteq B$ implies $(A] \subseteq (B]$; $((A]) = (A]$. Let M be a po - Γ -semigroup. A subset A of M is called a *left* (resp. *right*) *ideal* of M if (1) $M\Gamma A \subseteq A$ and (2) if $a \in A$ and $M \ni b \leq a$, then $b \in A$. It is called an *ideal* of M if it is both a left and right ideal of M . For an element a of M , we denote by $L(a)$, $R(a)$ and $I(a)$ the left ideal, right ideal and the ideal of M ,

respectively, generated by a , and we have $L(a) = (a \cup M\Gamma a]$, $R(a) = (a \cup a\Gamma M]$, and $I(a) = (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]$. We denote by \mathcal{L} the equivalence relation on M defined by $a\mathcal{L}b$ if and only if $L(a) = L(b)$, by \mathcal{R} the equivalence relation on M defined by $a\mathcal{R}b$ if and only if $R(a) = R(b)$ and by \mathcal{I} the equivalence relation on M defined by $a\mathcal{I}b$ if and only if $I(a) = I(b)$. A nonempty subset A of M is called a *subsemigroup* of M if for every $a, b \in A$ and every $\gamma \in \Gamma$ we have $a\gamma b \in A$, that is, if $A\Gamma A \subseteq A$. A subsemigroup F of M is called a *filter* of M if (1) $a, b \in F$ and $\gamma \in \Gamma$ such that $a\gamma b \in F$ implies $a \in F$ and $b \in F$ and (2) if $a \in F$ and $M \ni c \geq a$, then $c \in F$. An equivalence relation σ on M is called *congruence* if $(a, b) \in \sigma$ implies $(a\gamma c, b\gamma c) \in \sigma$ and $(c\gamma a, c\gamma b) \in \sigma$ for any $c \in M$ and any $\gamma \in \Gamma$. A congruence σ on M is called *semilattice congruence* if $(a\gamma b, b\gamma a) \in \sigma$ and $(a\gamma a, a) \in \sigma$ for every $a, b \in M$ and every $\gamma \in \Gamma$. If σ is a semilattice congruence on M , then the σ -class $(a)_\sigma$ of M containing the element a is a subsemigroup of M for every $a \in M$. A semilattice congruence σ on M is called *complete* if $a \leq b$ implies $(a, a\gamma b) \in \sigma$ for every $\gamma \in \Gamma$. We denote by \mathcal{N} the relation on M defined by $a\mathcal{N}b$ if and only if the filters of M generated by the elements a and b of M coincide. As in semigroups, the relation \mathcal{N} is a semilattice congruence on M . So, if $z \in M$ and $\gamma \in \Gamma$, then we have $(z\gamma z, z) \in \mathcal{N}$, $(z\gamma z\gamma z, z\gamma z) \in \mathcal{N}$, $(z\gamma z\gamma z\gamma z, z\gamma z\gamma z) \in \mathcal{N}$ and so on. In particular, exactly as in ordered semigroups, the relation \mathcal{N} is a complete semilattice congruence on M . A Γ -semigroup M is called *left* (resp. *right*) *simple* if for every left (resp. right) ideal T of M we have $T = M$, that is, if M is the only left (resp. right) ideal of M . A Γ -semigroup M is called *simple* if M is the only ideal of M . A po - Γ -semigroup M is said to be a *semilattice of simple* (resp. *left simple*) *semigroups* if there exists a semilattice congruence σ on M such that the σ -class $(x)_\sigma$ is a simple (resp. left simple) subsemigroup of M for every $x \in M$. A po - Γ -semigroup M is called a *chain of simple* (resp. *left simple*) *semigroups* if there exists a semilattice congruence σ on M such that $(x)_\sigma$ is a simple (resp. left simple) subsemigroup of M for every $x \in M$, and for any $x, y \in M$ and any $\gamma \in \Gamma$ we have $(x)_\sigma = (x\gamma y)_\sigma$ or $(y)_\sigma = (x\gamma y)_\sigma$.

Many results on Γ -semigroups can be obtained from semigroups just putting a “Gamma” in the appropriate place. But there are also results for which the transfer is not so easy. In a Γ -semigroup M the filter of M generated by an

element a of M plays an essential role in the structure, in particular, in the decomposition of some types of Γ -semigroups. So it is important to get the form of its elements. The existed definition of intra-regularity in the bibliography used to be the following: A po - Γ semigroup M is intra-regular if, by definition, $(a \in M\Gamma a\Gamma a\Gamma M]$ for every $a \in M$. With this definition is not possible to describe the form of the elements of the $N(x)$ ($x \in S$). To overcome this difficulty, the following new concept of intra-regularity has been introduced in [6]: We say that a po - Γ -semigroup M is intra-regular if $a \in (M\Gamma a\gamma a\Gamma M]$ for every $a \in M$. Using this definition, we first give a structure theorem referring to the decomposition of a po - Γ -semigroup into simple components. Then, for a po - Γ -semigroup M , we prove the following: The ideals of M are prime if and only if they form a chain and M is intra-regular. M is intra-regular and the ideals of M form a chain if and only if M is a chain of simple semigroups. For an intra-regular po - Γ -semigroup M the set $\{(x)_{\mathcal{N}} \mid x \in M\}$ coincides with the set of all maximal simple subsemigroups of M . In our investigation, we use the fact that if the ideals of a po - Γ -semigroup are weakly prime, then they form a chain. A po - Γ -semigroup M is called *left* (resp. *right*) *duo* if the left (resp. right) ideals of M are two-sided. Keeping the new definition of left (right) regularity of po - Γ -semigroups introduced in [6], we also give a structure theorem related with the decomposition of a po - Γ -semigroup M which is left regular and left duo into left simple components. If we want to get a result on a po - Γ -semigroup, we never work directly on the po - Γ -semigroup. Exactly as in the hypersemigroups, we have to solve it first for an ordered semigroup and then to be careful to define the analogous concepts in case of po - Γ -semigroups (if they do not defined directly) and put the “ Γ ” in the appropriate place. We never solve the problem directly in po - Γ -semigroups. The results of this paper are based on the corresponding results on ordered semigroups considered in [2] and [3], and the aim of writing this paper is to show the importance of these new concepts of intra-regularity and left (right) regularity considered in [6], in the investigation.

2 Main results

Let M be a po - Γ -semigroup. A subset A of M is called *idempotent*, if $A = (A\Gamma A]$. A subset T of M is called *prime* if $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$ implies $a \in T$ or $b \in T$. The set T is called *semiprime* if $a \in M$ and $\gamma \in \Gamma$ such that $a\gamma a \in T$ implies $a \in T$ [6]. A subset T of M is called *weakly prime* if the following assertion is satisfied:

If A, B are ideals of M such that $A\Gamma B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.

For a subset T of M , we consider the statements:

- (1) $a, b \in M, \gamma \in \Gamma, a\gamma b \in T \implies a \in T \text{ or } b \in T$.
- (2) $A, B \subseteq M, A\Gamma B \subseteq T \implies A \subseteq T \text{ or } B \subseteq T$.

Then (1) \implies (2). In fact: Let $A, B \subseteq M, A\Gamma B \subseteq T, A \not\subseteq T$ and $b \in B$. Take an element $a \in A$ such that $a \notin T$ and an element $\gamma \in \Gamma$ ($\Gamma \neq \emptyset$). Since $a\gamma b \in A\Gamma B \subseteq T$, by (1), we have $a \in T$ or $b \in T$. Since $a \notin T$, we get $b \in T$.

We have the following:

- (a) If T is a prime subset of M , then T is a semiprime subset of M .
- (b) If T is a prime subset of M , then T is a weakly prime subset of M .

Definition 1. [6] A po - Γ -semigroup M is called *intra-regular* if

$$x \in (M\Gamma x\gamma x\Gamma M]$$

for every $x \in M$ and every $\gamma \in \Gamma$.

Proposition 2. If M is an intra-regular po - Γ -semigroup then, for every $x, y \in M$ and every $\gamma \in \Gamma$, we have $(M\Gamma x\gamma y\Gamma M] = (M\Gamma y\gamma x\Gamma M]$.

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. Since $x\gamma y \in M\Gamma M \subseteq M$ and M is intra-regular, we have $x\gamma y \in (M\Gamma(x\gamma y)\gamma(x\gamma y)\Gamma M] \subseteq (M\Gamma y\gamma x\Gamma M]$. Then we have

$$\begin{aligned} M\Gamma(x\gamma y)\Gamma M &\subseteq (M]\Gamma(M\Gamma y\gamma x\Gamma M]\Gamma(M] \subseteq (M\Gamma M\Gamma y\gamma x\Gamma M\Gamma M] \\ &\subseteq (M\Gamma y\gamma x\Gamma M], \end{aligned}$$

from which $(M\Gamma(x\gamma y)\Gamma M] \subseteq ((M\Gamma y\gamma x\Gamma M]) = (M\Gamma y\gamma x\Gamma M]$. Since M is intra-regular and $y\gamma x \in M$, by symmetry, we get $(M\Gamma y\gamma x\Gamma M] \subseteq (M\Gamma x\gamma y\Gamma M]$, thus we have $(M\Gamma x\gamma y\Gamma M] = (M\Gamma y\gamma x\Gamma M]$. \square

Lemma 3. [6] *A po- Γ -semigroup M is intra-regular if and only if, for every $x \in M$, we have*

$$N(x) = \{y \in M \mid x \in (M\Gamma y\Gamma M)\}.$$

In a similar way as in [1] and [4], we can prove the following lemma.

Lemma 4. *If M is a po- Γ -semigroup, then $\mathcal{I} \subseteq \mathcal{N}$ and $\mathcal{I} \subseteq \mathcal{L}$.*

Lemma 5. [6] *A po- Γ -semigroup M is intra-regular if and only if the ideals of M are semiprime.*

The proof of the following lemma is easy.

Lemma 6. *If M is a po- Γ -semigroup, then the set $(M\Gamma a\Gamma M]$ (resp. $(M\Gamma a]$) is an ideal (resp. left ideal) of M , and the set $(a\Gamma M]$ is a right ideal of M for every $a \in M$.*

Definition 7. A po- Γ -semigroup S is said to be a *semilattice of simple* (resp. *left simple*) *semigroups* if there exists a semilattice congruence σ on M such that the σ -class $(x)_\sigma$ of M containing x is a simple (resp. left simple) subsemigroup of M for every $x \in M$.

Theorem 8. *Let M be an po- Γ -semigroup. The following are equivalent:*

- (1) *M is intra-regular.*
- (2) *$N(x) = \{y \in M \mid x \in (M\Gamma y\Gamma M)\}$ for every $x \in M$.*
- (3) *$\mathcal{N} = \mathcal{I}$.*
- (4) *For every ideal I of M , we have $I = \bigcup_{x \in I} (x)_\mathcal{N}$.*
- (5) *$(x)_\mathcal{N}$ is a simple subsemigroup of M for every $x \in M$.*
- (6) *M is a semilattice of simple semigroups.*
- (7) *Every ideal of M is semiprime.*

Proof. The implication (1) \Rightarrow (2) follows from Lemma 3, the proof of (3) \Rightarrow (4) is similar with the corresponding result for semigroups without order in [6], (5) \Rightarrow (6) since \mathcal{N} is a semilattice congruence on M and (7) \Rightarrow (1) by Lemma 5. (2) \Rightarrow (3). Let $(a, b) \in \mathcal{N}$. Since $a \in N(a) = N(b)$, by (2), we have $b \in (M\Gamma a\Gamma M] \subseteq (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M] = I(a)$, so $I(b) \subseteq I(a)$. Since $b \in N(a)$, by symmetry, we get $I(a) \subseteq I(b)$, so $I(a) = I(b)$, and $(a, b) \in \mathcal{I}$. Then $\mathcal{N} \subseteq \mathcal{I}$, on the other hand by Lemma 4, we have $\mathcal{I} \subseteq \mathcal{N}$, thus $\mathcal{I} = \mathcal{N}$.

(4) \implies (5). Let $x \in M$. Since \mathcal{N} is a semilattice congruence on M , $(x)_{\mathcal{N}}$ is a subsemigroup of M . Let I be an ideal of $(x)_{\mathcal{N}}$. Then $I = (x)_{\mathcal{N}}$. Indeed: Let $y \in (x)_{\mathcal{N}}$. Take an element $z \in I$ and an element $\gamma \in \Gamma$ ($\Gamma \neq \emptyset$). Since $(M\Gamma z\gamma z\gamma z\Gamma M]$ is an ideal of M (see Lemma 6), by hypothesis, we have $(M\Gamma z\gamma z\gamma z\Gamma M] = \bigcup_{t \in (M\Gamma z\gamma z\gamma z\Gamma M]} (t)_{\mathcal{N}}$. Since $y \in (x)_{\mathcal{N}} = (z)_{\mathcal{N}} = (z\gamma z\gamma z\gamma z\gamma z)_{\mathcal{N}} \subseteq (M\Gamma z\gamma z\gamma z\Gamma M]$, we have

$$y \leq a\delta z\gamma z\gamma z\xi b = (a\delta z)\gamma z\gamma(z\xi b) \text{ for some } a, b \in M, \delta, \xi \in \Gamma.$$

Using the fact that \mathcal{N} is a complete semilattice congruence on M , in a similar way as in [5], we prove that $a\delta z \in (x)_{\mathcal{N}}$ and $z\xi b \in (x)_{\mathcal{N}}$. Then, since I is an ideal of $(x)_{\mathcal{N}}$ and $z \in I$, we have $(a\delta z)\gamma z\gamma(z\xi b) \in (x)_{\mathcal{N}}\Gamma I\Gamma(x)_{\mathcal{N}} \subseteq I$, and $y \in I$. Hence $(x)_{\mathcal{N}} \subseteq I$, and so $I = (x)_{\mathcal{N}}$.

(6) \implies (7). Let σ be a semilattice congruence on M such that $(x)_{\sigma}$ is a simple subsemigroup of M for every $x \in M$. Let I be an ideal of M , $x \in M$ and $\gamma \in \Gamma$ such that $x\gamma x \in I$. The set $I \cap (x)_{\sigma}$ is an ideal of $(x)_{\sigma}$. Indeed: Taking into account the proof of the implication (6) \Rightarrow (7) in [5], it is enough to prove the following: Let $a \in I \cap (x)_{\sigma}$ and $(x)_{\sigma} \ni b \leq a$, then $b \in I \cap (x)_{\sigma}$. Since $M \ni b \leq a \in I$ and I is an ideal of M , we have $b \in I$, then $b \in I \cap (x)_{\sigma}$. Since $(x)_{\sigma}$ is a simple subsemigroup of M , we have $I \cap (x)_{\sigma} = (x)_{\sigma}$, then $x \in I$. Thus M is semiprime. \square

A subset A of a Γ -semigroup M is called *idempotent* if $A = (A\Gamma A]$.

Lemma 9. *Let M be a po- Γ -semigroup. The ideals of M are idempotent if and only if for any ideals A, B of M , we have $A \cap B = (A\Gamma B]$.*

Proof. \implies . Let A, B be ideals of M . Then $(A\Gamma B] \subseteq (A\Gamma M] \subseteq (A] = A$ and $(A\Gamma B] \subseteq (M\Gamma B] \subseteq (B] = B$, thus $(A\Gamma B] \subseteq A \cap B$. On the other hand, $A \cap B$ is an ideal of M . Indeed: Take an element $a \in A$, an element $b \in B$ and an element $\gamma \in \Gamma$ ($A, B, \Gamma \neq \emptyset$). Then $a\gamma b \in A\Gamma B \subseteq A\Gamma M \subseteq A$ and $a\gamma b \in A\Gamma B \subseteq M\Gamma B \subseteq B$, so $a\gamma b \in A \cap B$, and $\emptyset \neq A \cap B \subseteq M$. We also have $(A \cap B)\Gamma M \subseteq A\Gamma M \subseteq A$, $M\Gamma(A \cap B) \subseteq M\Gamma B \subseteq B$, and if $x \in A \cap B$ and $M \ni y \leq x$ then, since $x \in A$ we have $y \in A$ and since $x \in B$ we have $y \in B$, so $y \in A \cap B$. Since $A \cap B$ is an ideal of M , by hypothesis, we have $A \cap B = \left((A \cap B)\Gamma(A \cap B) \right] \subseteq (A\Gamma B]$. Hence we have $A \cap B = (A\Gamma B]$.

\Leftarrow . Let A be an ideal of M . By hypothesis, we have $A = (A\Gamma A]$, so A is idempotent. \square

Theorem 10. *Let M be a po- Γ -semigroup. The ideals of M are weakly prime if and only if they are idempotent and they form a chain.*

Proof. \Rightarrow . Let A, B be ideals of M . One can easily prove that $(A\Gamma B]$ is an ideal of M . Since $A, B, (A\Gamma B]$ are ideals of M , $A\Gamma B \subseteq (A\Gamma B]$ and $(A\Gamma B]$ is weakly prime, we have $A \subseteq (A\Gamma B] \subseteq (M\Gamma B] \subseteq (B] = B$ or $B \subseteq (A\Gamma B] \subseteq (A\Gamma M] \subseteq (A] = A$, thus the ideals of M form a chain. Furthermore, since A and $(A\Gamma A]$ are ideals of M , $A\Gamma A \subseteq (A\Gamma A]$ and $(A\Gamma A]$ is weakly prime, we have $A \subseteq (A\Gamma A] \subseteq (M\Gamma A] \subseteq (A] = A$, so $A = (A\Gamma A]$.

\Leftarrow . Let A, B, T be ideals of M such that $A\Gamma B \subseteq T$. If $A \subseteq B$ then, by Lemma 9, $A = A \cap B = (A\Gamma B] \subseteq (T] = T$. If $B \subseteq A$, then $B = A \cap B = (A\Gamma B] \subseteq (T] = T$. \square

Lemma 11. *Let M be a po- Γ -semigroup. If M is intra-regular, then*

$$I(x) = (M\Gamma x\Gamma M] \text{ for every } x \in M.$$

Proof. Let $x \in M$. Since $(M\Gamma x\Gamma M]$ is an ideal of M , by Lemma 5, it is semiprime. Take an element $\gamma \in \Gamma$ ($\Gamma \neq \emptyset$). Since $x\gamma x \in M$ and $(x\gamma x)\gamma(x\gamma x) \in (M\Gamma x\Gamma x]$, we have $x\gamma x \in (M\Gamma x\Gamma x]$, then $x \in (M\Gamma x\Gamma x]$, and $I(x) \subseteq (M\Gamma x\Gamma x]$. On the other hand, $(M\Gamma x\Gamma x] \subseteq I(x)$, thus we get $I(x) = (M\Gamma x\Gamma M]$. \square

Lemma 12. *If M is an po- Γ -semigroup, $x, y \in M$ and $\gamma \in \Gamma$, then*

$$I(x\gamma y) \subseteq I(x) \cap I(y).$$

In particular, if M is intra-regular, then $I(x\gamma y) = I(x) \cap I(y)$.

Proof. Since $x\gamma y \in I(x)\Gamma M \subseteq I(x)$ and $x\gamma y \in M\Gamma I(y) \subseteq I(y)$, we have $I(x\gamma y) \subseteq I(x) \cap I(y)$. Let now M be intra-regular and $t \in I(x) \cap I(y)$. Then, by Lemma 11, we have $t \in (M\Gamma x\Gamma M]$ and $t \in (M\Gamma y\Gamma M]$. Thus we have

$$t \leq a\mu x\rho b \text{ and } t \leq c\xi y\zeta d \text{ for some } a, b, c, d \in M, \mu, \rho, \xi, \zeta \in \Gamma.$$

Then $t\gamma t \leq (c\xi y\zeta d)\gamma(a\mu x\rho b) = c\xi(y\zeta d\gamma a\mu x)\rho b$. In addition, we have $y\zeta d\gamma a\mu x \in I(x\gamma y)$. Indeed, by Lemma 11,

$$(y\zeta d\gamma a\mu x)\gamma(y\zeta d\gamma a\mu x) \in M\Gamma(x\gamma y)\Gamma M \subseteq (M\Gamma(x\gamma y)\Gamma M] = I(x\gamma y).$$

Since M is intra-regular and $I(x\gamma y)$ is an ideal of M , by Lemma 5, $I(x\gamma y)$ is semiprime. So we get $y\zeta d\gamma a\mu x \in I(x\gamma y)$. Since $I(x\gamma y)$ is an ideal of M , we have $c\xi(y\zeta d\gamma a\mu x)\rho b \in M\Gamma I(x\gamma y)\Gamma M \subseteq I(x\gamma y)\Gamma M \subseteq I(x\gamma y)$, then $t\gamma t \in I(x\gamma y)$. Since $I(x\gamma y)$ is semiprime, we have $t \in I(x\gamma y)$. Thus we get $I(x) \cap I(y) \subseteq I(x\gamma y)$ and so $I(x\gamma y) = I(x) \cap I(y)$. \square

Theorem 13. *Let M be a po- Γ -semigroup. The ideals of M are prime if and only if they form a chain and M is intra-regular.*

Proof. \implies . The ideals of M are prime, so they are weakly prime and semiprime. Since they are weakly prime, by Theorem 10, they form a chain. Let now $a \in M$ and $\gamma \in \Gamma$. Since $(M\Gamma a\gamma a\Gamma M]$ is an ideal of M , $a\gamma a \in (M\Gamma a\gamma a\Gamma M]$ and $(M\Gamma a\gamma a\Gamma M]$ is semiprime, we have $a\gamma a \in (M\Gamma a\gamma a\Gamma M]$, and $a \in (M\Gamma a\gamma a\Gamma M]$. Thus M is intra-regular.

\Leftarrow . Suppose M is intra-regular and the ideals of M form a chain. Let now T be an ideal of M , $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$. We have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$ then, by Lemma 12, we have $a \in I(a) = I(a) \cap I(b) = I(a\gamma b) \subseteq I(T) = T$. If $I(b) \subseteq I(a)$, then $b \in I(b) = I(a) \cap I(b) = I(a\gamma b) \subseteq T$. \square

Proposition 14. *Let M be an intra-regular po- Γ -semigroup. If the ideals of M form a chain then, for every $x, y \in M$ and every $\gamma \in \Gamma$, we have*

$$x \in (M\Gamma x\gamma y\Gamma M] \text{ or } y \in (M\Gamma x\gamma y\Gamma M].$$

Proof. Assuming the ideals of M form a chain, let $x, y \in M$ and $\gamma \in \Gamma$. By Theorem 13, the ideals of M are prime. Since $(M\Gamma x\gamma y\Gamma M]$ is an ideal of M , $(M\Gamma x\gamma y\Gamma M]$ is prime. Since $(x\gamma x)\gamma(y\gamma y) \in (M\Gamma x\gamma y\Gamma M]$, we have $x\gamma x \in (M\Gamma x\gamma y\Gamma M]$ or $y\gamma y \in (M\Gamma x\gamma y\Gamma M]$. If $x\gamma x \in (M\Gamma x\gamma y\Gamma M]$ then, since $(M\Gamma x\gamma y\Gamma M]$ is prime, we have $x \in (M\Gamma x\gamma y\Gamma M]$. If $y\gamma y \in (M\Gamma x\gamma y\Gamma M]$, then $y \in (M\Gamma x\gamma y\Gamma M]$. \square

Definition 15. A po- Γ -semigroup M is called a *chain of simple semigroups* if there exists a semilattice congruence σ on M such that $(x)_\sigma$ is a simple sub-semigroup of M for every $x \in M$, and for any $x, y \in M$ and any $\gamma \in \Gamma$ we have

$$(x)_\sigma = (x\gamma y)_\sigma \text{ or } (y)_\sigma = (x\gamma y)_\sigma$$

(in other words, the set M/σ endowed with the relation

$$(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (x\gamma y)_\sigma \quad \forall \gamma \in \Gamma$$

is a chain).

Theorem 16. *A po- Γ -semigroup M is intra-regular and the ideals of M form a chain if and only if M is chain of simple semigroups.*

Proof. \Rightarrow . Since M is intra-regular and \mathcal{N} is a semilattice congruence on M , by Theorem 8(1) \Rightarrow (5), $(x)_\mathcal{N}$ is a simple subsemigroup of M for every $x \in M$, so M is a semilattice of simple semigroups. Let now $x, y \in M$ and $\gamma \in \Gamma$. By Proposition 14, we have $x \in (M\Gamma x\gamma y\Gamma M]$ or $y \in (M\Gamma x\gamma y\Gamma M]$. If $x \in (M\Gamma x\gamma y\Gamma M]$, then

$$N(x) \ni x \leq a\mu(x\gamma y)\rho b \text{ for some } a, b \in M, \mu, \rho \in \Gamma.$$

Since $N(x)$ is a filter of M , we have $a\mu(x\gamma y)\rho b \in N(x)$, $x\gamma y \in N(x)$ and $N(x\gamma y) \subseteq N(x)$. If $y \in (M\Gamma x\gamma y\Gamma M]$, then

$$N(y) \ni y \leq c\xi(x\gamma y)\zeta d \text{ for some } c, d \in M, \xi, \zeta \in \Gamma,$$

then $c\xi(x\gamma y)\zeta d \in N(y)$, $x\gamma y \in N(y)$, and $N(x\gamma y) \subseteq N(y)$. On the other hand, since $x\gamma y \in N(x\gamma y)$, we have $x \in N(x\gamma y)$ and $y \in N(x\gamma y)$, so $N(x) \subseteq N(x\gamma y)$ and $N(y) \subseteq N(x\gamma y)$. Thus we get $N(x) = N(x\gamma y)$ or $N(y) = N(x\gamma y)$, then $(x)_\mathcal{N} = (x\gamma y)_\mathcal{N}$ or $(y)_\mathcal{N} = (x\gamma y)_\mathcal{N}$. Therefore M is a chain of simple semigroups.

\Leftarrow . Let σ be a semilattice congruence on M such that $(x)_\sigma$ is a simple subsemigroup of M for every $x \in M$ and $(M/\sigma, \preceq)$ is a chain. By Theorem 13 it is enough to prove that the ideals of M are prime. So, let I be an ideal of M , $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in I$. The set $(a\gamma b)_\sigma \cap I$ is an ideal of $(a\gamma b)_\sigma$. Indeed:

$$\emptyset \neq (a\gamma b)_\sigma \cap I \subseteq (a\gamma b)_\sigma \quad (a\gamma b \in (a\gamma b)_\sigma, a\gamma b \in I)$$

$$\begin{aligned} ((a\gamma b)_\sigma \cap I)\Gamma(a\gamma b)_\sigma &\subseteq (a\gamma b)_\sigma \Gamma(a\gamma b)_\sigma \cap I\Gamma(a\gamma b)_\sigma = (a\gamma b)_\sigma \cap I\Gamma(a\gamma b)_\sigma \\ &\subseteq (a\gamma b)_\sigma \cap I\Gamma M \subseteq (a\gamma b)_\sigma \cap I, \end{aligned}$$

$$\begin{aligned} (a\gamma b)_\sigma \Gamma((a\gamma b)_\sigma \cap I) &\subseteq (a\gamma b)_\sigma \Gamma(a\gamma b)_\sigma \cap (a\gamma b)_\sigma \Gamma I \subseteq (a\gamma b)_\sigma \cap M\Gamma I \\ &\subseteq (a\gamma b)_\sigma \cap I. \end{aligned}$$

Let $x \in (a\gamma b)_\sigma \cap I$ and $(a\gamma b)_\sigma \ni y \leq x$. Since $M \ni y \leq x \in I$ and I is an ideal of M , we have $y \in I$, then $y \in (a\gamma b)_\sigma \cap I$. Since $(a\gamma b)_\sigma$ is simple, we have $(a\gamma b)_\sigma \cap I = (a\gamma b)_\sigma$. By hypothesis, $(a)_\sigma = (a\gamma b)_\sigma$ or $(b)_\sigma = (a\gamma b)_\sigma$. Then we have $a \in I$ or $b \in I$, thus I is a prime. \square

Lemma 17. *Let M be an po- Γ -semigroup, T a subsemigroup of M and $x \in T$. Then the set $(M\Gamma x\Gamma M] \cap T$ is an ideal of T .*

Proof. The set $(M\Gamma x\Gamma M] \cap T$ is a nonempty subset of M . This is because, for any $\gamma \in \Gamma$, $x\gamma x\gamma x \in M\Gamma x\Gamma M$ and $x\gamma x\gamma x \in (T\Gamma T)\Gamma T \subseteq T\Gamma T \subseteq T$. Moreover,

$$\begin{aligned} \left((M\Gamma x\Gamma M] \cap T \right) \Gamma T &\subseteq (M\Gamma x\Gamma M] \Gamma T \cap T\Gamma T \subseteq (M\Gamma x\Gamma M] \Gamma (M] \cap T \\ &\subseteq \left(M\Gamma x\Gamma (M\Gamma M) \right] \cap T \subseteq (M\Gamma x\Gamma M] \cap T, \end{aligned}$$

In a similar way, we have $T\Gamma \left((M\Gamma x\Gamma M] \cap T \right) \subseteq (M\Gamma x\Gamma M] \cap T$. Let now $a \in (M\Gamma x\Gamma M] \cap T$ and $T \ni b \leq a$. Since $a \in (M\Gamma x\Gamma M]$, there exist $u, v \in M$ and $\xi, \zeta \in \Gamma$ such that $a \leq u\xi x\zeta v$. Then we have $b \leq u\xi x\zeta v \in M\Gamma x\Gamma M$, and $b \in (M\Gamma x\Gamma M]$. \square

Theorem 18. *Let M be an intra-regular po- Γ -semigroup. Then the set $(x)_\mathcal{N}$ is a maximal simple subsemigroup of M for every $x \in M$. Conversely, if T is a maximal simple subsemigroup of M , then there exists $x \in M$ such that $T = (x)_\mathcal{N}$.*

Proof. \implies . Let $x \in M$. By the Theorem 8(1) \Rightarrow (5), the set $(x)_\mathcal{N}$ is a simple subsemigroup of M . Let now T be a simple subsemigroup of M such that $T \supseteq (x)_\mathcal{N}$. Then $T = (x)_\mathcal{N}$. Indeed: Let $y \in T$. Since $x \in T$, by Lemma 17, the set $(M\Gamma x\Gamma M] \cap T$ is an ideal of T . Since T is a simple, we have $(M\Gamma x\Gamma M] \cap T = T$, then $y \in (M\Gamma x\Gamma M]$. Since M is intra-regular, by Lemma 3, we have $x \in N(y)$, and $N(x) \subseteq N(y)$. On the other hand, since $y \in T$, the set $(M\Gamma y\Gamma M] \cap T$ is an ideal of T . So $(M\Gamma y\Gamma M] \cap T = T$, $x \in (M\Gamma y\Gamma M]$, $y \in N(x)$, and $N(y) \subseteq N(x)$. Therefore we have $N(x) = N(y)$, so $y \in (x)_\mathcal{N}$, and $T \subseteq (x)_\mathcal{N}$. Since $(x)_\mathcal{N}$ is a subsemigroup of M , we get $T = (x)_\mathcal{N}$.

\impliedby . Let T be a maximal simple subsemigroup of M . Take an element $x \in T$ ($T \neq \emptyset$). Exactly as in the proof of the “ \implies ”-part of the theorem given above, we prove that $T \subseteq (x)_\mathcal{N}$. Since $(x)_\mathcal{N}$ is a simple subsemigroup of M (cf. Theorem 8(1) \Rightarrow (5)) and T is a maximal simple subsemigroup of M , we have $T = (x)_\mathcal{N}$.

□

Corollary 19. *For an intra-regular po- Γ -semigroup M , the set $\{(x)_{\mathcal{N}} \mid x \in M\}$ coincides with the set of all maximal simple subsemigroup of M .*

Definition 20. [6] A Γ -semigroup M is called *left* (resp. *right*) *regular* if

$$x \in (M\Gamma x\gamma x] \text{ (resp. } x \in (x\gamma x\Gamma M])$$

for every $x \in M$ and every $\gamma \in \Gamma$.

In a similar way as in Theorem 8 above and the Theorem 6 in [6], we can prove the following

Theorem 21. *Let M be a po- Γ -semigroup. The following are equivalent:*

- (1) M is left regular and left duo.
- (2) $N(x) = \{y \in M \mid x \in (M\Gamma y)\}$ for every $x \in M$.
- (3) $\mathcal{N} = \mathcal{L}$.
- (4) For every left ideal L of M , we have $L = \bigcup_{x \in L} (x)_{\mathcal{N}}$.
- (5) $(x)_{\mathcal{N}}$ is a left simple subsemigroup of M for every $x \in M$.
- (6) M is a semilattice of left simple semigroups.
- (7) Every left ideal of M is semiprime and two-sided.

The right analogue of the above theorem also holds.

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University of Athens, Department of Mathematics, 15784 Panepistimiopolis, Greece
email: nkehayop@math.uoa.gr